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順序を保存する作用素不等式のある拡張などについて
On further extensions of order preserving operator inequality

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1. Introduction

Each capital letter means a bounded linear operator on a Hilbert space. An operator T is said to be positive semidefinite (denoted by $0 \leq T$) if $0 \leq (Tx, x)$ for all vectors x.

The readers should pay attention to that the statements cited here might be neither the precise repetition of nor the full strength as in their original articles.

Theorem (Löwner-Heinz).

Let $0 \leq p \leq 1$. $0 \leq B \leq A$

$\Longrightarrow$

$$B^p \leq A^p.$$

It is well-known that for $1 < p$, $0 \leq B \leq A$ does not always ensure $B^p \leq A^p$.

Example. Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $0 \leq B \leq A$ and $B^2 \not\leq A^2$.

Therefore, $0 \le B \le A$ does not always imply $AB^2A \le A^4$. (Consider multiplying A^{-1} from both sides.)

Conjecture (Chan and Kwong '85).

$0 \le B \le A$

$\overset{?}{\Longrightarrow}$

$$(AB^2A)^{\frac{1}{2}} \le A^2.$$

The Furuta inequality was epochmaking on this direction.

Theorem (Furuta '87).

Let $0 \le p,\ 1 \le q,\ 0 \le r$ and $p + r \le (1+r)q$.

$0 \le B \le A$

$\Longrightarrow$

$$\left(A^{\frac{r}{2}}\, B^p\, A^{\frac{r}{2}}\right)^{\frac{1}{q}} \le A^{\frac{p+r}{q}}.$$

Remark.

- $r = 0$: Löwner-Heinz
- $p = q = r = 2$: Chan-Kwong's conjecture

(essential case)

Let $1 \le p,\ 0 \le r$.

$0 \le B \le A$

$\Longrightarrow$

$$\left(A^{\frac{r}{2}}\, B^p\, A^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}} \le A^{1+r}.$$

Theorem (Ando and Hiai '94).

Let $1 \le p,\ \ 1 \le r.$

$0 \le B \le A$ and $\exists A^{-1}$

$\Longrightarrow$

$$\left\{A^{\frac{r}{2}} \left(A^{-\frac{1}{2}} B^p A^{-\frac{1}{2}}\right)^r A^{\frac{r}{2}}\right\}^{\frac{1}{p}} \le A^r.$$

Theorem (Furuta '95).

Let $1 \le p,\ \ 1 \le s,\ \ 0 \le t \le 1,\ \ t \le r.$

$0 \le B \le A$ and $\exists A^{-1}$

$\Longrightarrow$

$$\left\{A^{\frac{r}{2}} \left(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}}\right)^s A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)s+r}} \le A^{1-t+r}.$$

Remark.

- $t = 0,\ s = 1$: Furuta '87
- $t = 1,\ s = r$: Ando-Hiai

Theorem (Furuta '08).

Let $1 \le p_1, \cdots, p_{2n},\ 0 \le t \le 1,\ t \le r.$

$0 \le B \le A$ and $\exists A^{-1}$

$\Longrightarrow$

$$\left\{A^{\frac{r}{2}}\left(A^{-\frac{t}{2}}\cdots\left(A^{\frac{t}{2}}\left(A^{-\frac{t}{2}}B^{p_1}A^{-\frac{t}{2}}\right)^{p_2}A^{\frac{t}{2}}\right)^{p_3}\cdots A^{-\frac{t}{2}}\right)^{p_{2n}}A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{\varphi[2n;r,t]}}$$

$$\le A^{1-t+r}.$$

Remark. $n = 1$: Furuta '95

Definition.

$$\varphi[2n;r,t] = (\cdots(((p_1 - t)p_2 + t)\,p_3 - t)\,p_4 + \cdots - t)\,p_{2n} + r.$$

Up to here, we are concerned with 2 operators. The following theorem treats 3 operators.

Theorem (Uchiyama '03).

Let $1 \le p_1, p_2,\ 0 \le t_1 \le 1,\ t_1 \le t_2.$

$0 \le B \le A_1 \le A_2$ and $\exists A_1^{-1}$

$\Longrightarrow$

$$\left\{A_2^{\frac{t_2}{2}}\left(A_1^{-\frac{t_1}{2}}B^{p_1}A_1^{-\frac{t_1}{2}}\right)^{p_2}A_2^{\frac{t_2}{2}}\right\}^{\frac{1-t_1+t_2}{(p_1-t_1)p_2+t_2}} \le A_2^{1-t_1+t_2}.$$

Remark. $A_1 = A_2$: Furuta '95

Theorem (Yang and Wang '10).

$1 \leq p_1, \cdots, p_{2n},$

$0 \leq t_1, \cdots, t_n \leq 1, \;\; t_n \leq r,$

$0 \leq B \leq A_1 \leq A_2 \leq \cdots \leq A_{2n-1} \leq A_{2n}$ and $\exists A_1^{-1}$

$\Longrightarrow$

$$\Big\{ A_{2n}^{\frac{r}{2}} \Big(A_{2n-1}^{-\frac{t_n}{2}} \Big(A_{2n-2}^{\frac{t_{n-1}}{2}} \cdots A_4^{\frac{t_2}{2}} \Big[A_3^{-\frac{t_2}{2}} \Big\{ A_2^{\frac{t_1}{2}} \Big(A_1^{-\frac{t_1}{2}} B^{p_1} A_1^{-\frac{t_1}{2}} \Big)^{p_2} A_2^{\frac{t_1}{2}} \Big\}^{p_3} A_3^{-\frac{t_2}{2}} \Big]^{p_4} A_4^{\frac{t_2}{2}} \cdots A_{2n-2}^{\frac{t_{n-1}}{2}} \Big)^{p_{2n-1}} A_{2n-1}^{-\frac{t_n}{2}} \Big)^{p_{2n}} A_{2n}^{\frac{r}{2}} \Big\}^{\frac{1-t_n+r}{\eth[2n]-t_n+r}}$$

$\leq A_{2n}^{1-t_n+r}.$

Remark.

- $n = 1$: Uchiyama '03
- $A_1 = \cdots = A_{2n}$, $t_1 = \cdots = t_n$: Furuta'08

Definition.

$$\eth[2n] = \{ \cdots ((((p_1 - t_1)p_2 + t_1) p_3 - t_2) p_4 + t_2) p_5 - \cdots - t_n \} p_{2n} + t_n.$$

2. Some extensions of operator inequalities

Theorem 1(KW).

$1 \leq p_1, \cdots, p_{2n},$

$0 \leq t_{2k-1} \leq 1, \ \ t_{2k-1} \leq t_{2k} \, (k = 1, \cdots, n),$

$0 \leq B \leq A_1 \leq A_2, \ \ \exists A_1^{-1}$ and

$$A_{2k-2}^{\alpha(2k-2)} \leq A_{2k-1}^{\alpha(2k-2)} \leq A_{2k}^{\alpha(2k-2)} \quad (k = 2, \cdots n)$$

$$\Longrightarrow$$

$$\left\{ A_{2n}^{\frac{t_{2n}}{2}} \left(A_{2n-1}^{-\frac{t_{2n-1}}{2}} \cdots \left(A_2^{\frac{t_2}{2}} \left(A_1^{-\frac{t_1}{2}} B^{p_1} A_1^{-\frac{t_1}{2}} \right)^{p_2} A_2^{\frac{t_2}{2}} \right)^{p_3} \cdots A_{2n-1}^{-\frac{t_{2n-1}}{2}} \right)^{p_{2n}} A_{2n}^{\frac{t_{2n}}{2}} \right\}^{\frac{\alpha(2n)}{\psi(2n)}}$$

$$\leq A_{2n}^{\alpha(2n)}.$$

Definition.

$\alpha(2n) = 1 - t_1 + t_2 - \cdots - t_{2n-1} + t_{2n}$

$\psi(2n) = \{\cdots(((p_1 - t_1)p_2 + t_2)\, p_3 - t_3)\, p_4 + \cdots - t_{2n-1}\}\, p_{2n} + t_{2n}.$

Operator inequalities in the assumption of Theorem 1:

$$0 \leq B \leq A_1 \leq A_2$$

$$A_2^{1-t_1+t_2} \leq A_3^{1-t_1+t_2} \leq A_4^{1-t_1+t_2} \qquad (*)$$

$$A_4^{1-t_1+t_2-t_3+t_4} \leq A_5^{1-t_1+t_2-t_3+t_4} \leq A_6^{1-t_1+t_2-t_3+t_4}$$

$$\vdots$$

$$A_{2n-2}^{\alpha(2n-2)} \leq A_{2n-1}^{\alpha(2n-2)} \leq A_{2n}^{\alpha(2n-2)}.$$

For the condition $(*)$, it is sufficient that $A_2 \leq A_3 \leq A_4$ and

(i) $t_1 = t_2$

or

(ii) commute, especially $A_2 = A_3 = A_4$.

The full proof of Theorem 1 is just a mathematical induction. It is so natural and simple that one can understand the whole if he/she once see the proof for the case of $n = 2$.

Proof of Theorem 1 for $n = 2$.

Let $1 \leq p_1,\ p_2,\ p_3,\ p_4,$

$0 \leq t_1,\ t_3, \leq 1, \quad t_1 \leq t_2,\ \ t_3 \leq t_4,$

$0 \leq B \leq A_1 \leq A_2,\ \ \exists A_1^{-1}$ and

$$A_2^{1-t_1+t_2} \leq A_3^{1-t_1+t_2} \leq A_4^{1-t_1+t_2}.$$

By Uchiyama '03,

$$\left\{A_2^{\frac{t_2}{2}}\left(A_1^{-\frac{t_1}{2}} B^{p_1} A_1^{-\frac{t_1}{2}}\right)^{p_2} A_2^{\frac{t_2}{2}}\right\}^{\frac{1-t_1+t_2}{(p_1-t_1)p_2+t_2}} \leq A_2^{1-t_1+t_2}.$$

Denote the left hand side by B_1, then

$$B_1 \leq A_2^{1-t_1+t_2} \leq A_3^{1-t_1+t_2} \leq A_4^{1-t_1+t_2}.$$

Idea: Apply once again Uchiyama '03.

Put

$$p = \frac{(p_1 - t_1)p_2 + t_2}{1 - t_1 + t_2} p_3, \quad t = \frac{t_3}{1 - t_1 + t_2},$$
$$r = \frac{t_4}{1 - t_1 + t_2}, \quad s = p_4.$$

Then

$$1 \leq p,\ 1 \leq s,\ 0 \leq t \leq 1,\ t \leq r.$$

So we may apply Uchiyama '03 to

$$0 \le B_1 \le A_3^{1-t_1+t_2} \le A_4^{1-t_1+t_2},$$

which yields that

$$\left\{ \left(A_4^{1-t_1+t_2}\right)^{\frac{r}{2}} \left(\left(A_3^{1-t_1+t_2}\right)^{-\frac{t}{2}} B_1^p \left(A_3^{1-t_1+t_2}\right)^{-\frac{t}{2}} \right)^s \left(A_4^{1-t_1+t_2}\right)^{\frac{r}{2}} \right\}^{\frac{1-t+r}{(p-t)s+r}}$$

$$\le \left(A_4^{1-t_1+t_2}\right)^{1-t+r}.$$

At first,

$$(1-t_1+t_2)(1-t+r)$$

$$= (1-t_1+t_2)\left(1 - \frac{t_3}{1-t_1+t_2} + \frac{t_4}{1-t_1+t_2}\right)$$

$$= 1-t_1+t_2-t_3+t_4,$$

so the right hand side is $A_4^{1-t_1+t_2-t_3+t_4}$.

By the definition of B_1 and p,

$$B_1^p = \left(A_2^{\frac{t_2}{2}} \left(A_1^{-\frac{t_1}{2}} B^{p_1} A_1^{-\frac{t_1}{2}} \right)^{p_2} A_2^{\frac{t_2}{2}} \right)^{p_3}.$$

Obviously,

$$\left(A_3^{1-t_1+t_2}\right)^{-\frac{t}{2}} = A_3^{-\frac{t_3}{2}} \text{ and } \left(A_4^{1-t_1+t_2}\right)^{\frac{r}{2}} = A_4^{\frac{t_4}{2}}.$$

Moreover,

$$\frac{1-t+r}{(p-t)s+r}$$

$$= \frac{1-\frac{t_3}{1-t_1+t_2}+\frac{t_4}{1-t_1+t_2}}{\left(\frac{(p_1-t_1)p_2+t_2}{1-t_1+t_2}-\frac{t_3}{1-t_1+t_2}\right)p_4+\frac{t_4}{1-t_1+t_2}}$$

$$= \frac{1-t_1+t_2-t_3+t_4}{(((p_1-t_1)p_2+t_2)p_3-t_3)\,p_4+t_4}$$

$$= \frac{\alpha(4)}{\psi(4)}.$$

Thus we have

$$\left\{A_4^{\frac{t_4}{2}}\left(A_3^{-\frac{t_3}{2}}\left(A_2^{\frac{t_2}{2}}\left(A_1^{-\frac{t_1}{2}}B^{p_1}A_1^{-\frac{t_1}{2}}\right)^{p_2}A_2^{\frac{t_2}{2}}\right)^{p_3}A_3^{-\frac{t_3}{2}}\right)^{p_4}A_4^{\frac{t_4}{2}}\right\}^{\frac{\alpha(4)}{\psi(4)}}$$

$\leq A_4^{\alpha(4)}$. $\square$

Remark to Theorem 1.

We can't reduce the part of the assumption

$$A_{2k-2}^{\alpha(2k-2)} \leq A_{2k-1}^{\alpha(2k-2)} \leq A_{2k}^{\alpha(2k-2)} \quad (k = 2, \cdots n)$$

to

$$A_2 \leq \cdots \leq A_{2n}.$$

Easy counter example even in $n = 2$.

Take $I \leq C_1 \leq C_2$ such that $C_1^2 \not\leq C_2^2$.

Put

$$p_1 = \cdots = p_4 = 1,$$

$$t_1 = t_3 = 1, \quad t_2 = t_4 = 2,$$

$$B = A_1 = I, \quad A_2 = C_1, \quad A_3 = A_4 = C_2.$$

In this case, $\alpha(4) = \psi(4) = 3$.
If the inequality of the conclusion of Theorem 1 holds, we would have

$$C_2 C_2^{-\frac{1}{2}} C_1^2 C_2^{-\frac{1}{2}} C_2 \leq C_2^3,$$

which leads to $C_1^2 \leq C_2^2$, a contradiction.

Theorem 2(KW).

$1 \leq p_1, \cdots, p_{2n+1},$

$0 \leq t_1,\ 0 \leq t_{2k} \leq 1,\ t_{2k} \leq t_{2k+1}\ (k = 1, \cdots, n),$

$0 \leq B \leq A_1,\ \exists A_1^{-1}$ and

$$A_{2k-1}^{\beta(2k-1)} \leq A_{2k}^{\beta(2k-1)} \leq A_{2k+1}^{\beta(2k-1)} \quad (k = 1, \cdots n)$$

$\Longrightarrow$

$$\left\{ A_{2n+1}^{\frac{t_{2n+1}}{2}} \left(A_{2n}^{-\frac{t_{2n}}{2}} \cdots \left(A_2^{-\frac{t_2}{2}} \left(A_1^{\frac{t_1}{2}} B^{p_1} A_1^{\frac{t_1}{2}} \right)^{p_2} A_2^{-\frac{t_2}{2}} \right)^{p_3} \cdots A_{2n}^{-\frac{t_{2n}}{2}} \right)^{p_{2n+1}} A_{2n+1}^{\frac{t_{2n+1}}{2}} \right\}^{\frac{\beta(2n+1)}{\gamma(2n+1)}}$$

$$\leq A_{2n+1}^{\beta(2n+1)}.$$

Definition.

$$\beta(2n+1) = 1 + t_1 - t_2 + \cdots + t_{2n+1}$$

$$\gamma(2n+1) = \{\cdots((p_1 + t_1)p_2 - t_2)\, p_3 + \cdots - t_{2n}\}\, p_{2n+1} + t_{2n+1}.$$

Theorem 3(KW).

ℓ : even natural number, $1 \leq p_1, \cdots, p_{2n+\ell}$,

$0 \leq t_1, \cdots, t_n, t_{n+1}, t_{n+3}, \cdots, t_{n+\ell-1} \leq 1$,

$t_{n+1} \leq t_{n+2}, \cdots, t_{n+\ell-1} \leq t_{n+\ell}$,

$0 \leq B \leq A_1 \leq A_2 \leq \cdots \leq A_{2n+2}$ and $\exists A_1^{-1}$

$\Longrightarrow$

$$\Bigg\{ A_{2n+2}^{\frac{t_{n+\ell}}{2}} \Bigg(A_{2n+2}^{-\frac{t_{n+\ell-1}}{2}} \cdots \Bigg(A_{2n+2}^{\frac{t_{n+2}}{2}} \Bigg(A_{2n+1}^{-\frac{t_{n+1}}{2}} \Bigg(A_{2n}^{\frac{t_n}{2}}$$

$$\Bigg(A_{2n-1}^{-\frac{t_n}{2}} \cdots \Bigg(A_2^{\frac{t_1}{2}} \Bigg(A_1^{-\frac{t_1}{2}} B^{p_1} A_1^{-\frac{t_1}{2}} \Bigg)^{p_2} A_2^{\frac{t_1}{2}} \Bigg)^{p_3} \cdots A_{2n-1}^{-\frac{t_n}{2}} \Bigg)^{p_{2n}}$$

$$A_{2n}^{\frac{t_n}{2}} \Bigg)^{p_{2n+1}} A_{2n+1}^{-\frac{t_{n+1}}{2}} \Bigg)^{p_{2n+2}} A_{2n+2}^{\frac{t_{n+2}}{2}} \Bigg)^{p_{2n+3}} \cdots A_{2n+2}^{-\frac{t_{n+\ell-1}}{2}} \Bigg)^{p_{2n+\ell}} A_{2n+2}^{\frac{t_{n+\ell}}{2}} \Bigg\}^{\frac{\alpha'}{\psi'}}$$

$$\leq A_{2n+2}^{\alpha'},$$

where

$$\alpha' = 1 - t_{n+1} + t_{n+2} - \cdots - t_{n+\ell-1} + t_{n+\ell}$$

$$\psi' = (\cdots(((((p_1 - t_1)p_2 + t_1)\,p_3 - \cdots - t_n)\,p_{2n} + t_n)\,p_{2n+1} - t_{n+1})\,p_{2n+2} + \cdots - t_{n+\ell-1})\,p_{2n+\ell} + t_{n+\ell}.$$

Corollary.

$1 \leq p_1, \cdots, p_{2n},$

$0 \leq t_{2k-1} \leq 1, \ \ t_{2k-1} \leq t_{2k} \, (k = 1, \cdots, n),$

$0 \leq B \leq A$ and $\exists A^{-1}$

$\Longrightarrow$

$$\left\{ A^{\frac{t_{2n}}{2}} \left(A^{-\frac{t_{2n-1}}{2}} \cdots \left(A^{\frac{t_2}{2}} \left(A^{-\frac{t_1}{2}} B^{p_1} A^{-\frac{t_1}{2}} \right)^{p_2} A^{\frac{t_2}{2}} \right)^{p_3} \cdots A^{-\frac{t_{2n-1}}{2}} \right)^{p_{2n}} A^{\frac{t_{2n}}{2}} \right\}^{\frac{\alpha(2n)}{\psi(2n)}}$$

$$\leq A^{\alpha(2n)}.$$

Corollary.

$1 \leq p_1, \cdots, p_{2n+1},$

$0 \leq t_1, \ \ 0 \leq t_{2k} \leq 1, \ \ t_{2k} \leq t_{2k+1} \, (k = 1, \cdots, n),$

$0 \leq B \leq A$ and $\exists A_1^{-1}$

$\Longrightarrow$

$$\left\{ A^{\frac{t_{2n+1}}{2}} \left(A^{-\frac{t_{2n}}{2}} \cdots \left(A^{-\frac{t_2}{2}} \left(A^{\frac{t_1}{2}} B^{p_1} A^{\frac{t_1}{2}} \right)^{p_2} A^{-\frac{t_2}{2}} \right)^{p_3} \cdots A^{-\frac{t_{2n}}{2}} \right)^{p_{2n+1}} A^{\frac{t_{2n+1}}{2}} \right\}^{\frac{\beta(2n+1)}{\gamma(2n+1)}}$$

$$\leq A^{\beta(2n+1)}.$$

3. On range of parameters which make operator inequalities valid

Tanahashi showed the best possibility of the range in Furuta '87

$$p+r \leq (1+r)q \quad \text{and} \quad 1 \leq q$$

as far as one considers positive parameters.

Theorem (Tanahashi '96).

Let $0 < p,\ q,\ r$. $\quad (1+r)q < p+r \quad \text{or} \quad 0 < q < 1$

$\Longrightarrow \ \exists (A, B): \ 0 < B \leq A,$

$$\left(A^{\frac{r}{2}} B^{p} A^{\frac{r}{2}}\right)^{\frac{1}{q}} \not\leq A^{\frac{p+r}{q}}.$$

Corollary.

Let $1 \leq p,\ 0 \leq r$. $\quad 1 < \alpha$

$\Longrightarrow \ \exists (A, B): \ 0 < B \leq A,$

$$\left(A^{\frac{r}{2}} B^{p} A^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}\alpha} \not\leq A^{(1+r)\alpha}.$$

Corollary.

Let $0 < p < 1,\ 0 < r$.

$\Longrightarrow \ \exists (A, B): \ 0 < B \leq A,$

$$\left(A^{\frac{r}{2}} B^{p} A^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}} \not\leq A^{1+r}.$$

Tanahashi also obtained the best possibility of the outer power in the grand Furuta inequality.

Theorem (Tanahashi '99).

Let $1 \le p,\ 1 \le s,\ 0 \le t \le 1,\ t \le r.\quad 1 < \alpha$

$\Longrightarrow\ \exists (A, B):\ 0 < B \le A,$

$$\left\{A^{\frac{r}{2}}\left(A^{-\frac{t}{2}}B^{p}A^{-\frac{t}{2}}\right)^{s}A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)s+r}\alpha} \not\le A^{(1-t+r)\alpha}.$$

Theorem 4(KW).

Let $1 < p_1,\ 0 < p_j \le 1\,(j = 2, \cdots, 2n),$

$1 \le p_{2n+1},\ p_{2n+2},$

$0 \le t_j \le 1\,(j = 1, \cdots, n+1),\ t_{n+1} \le t_{n+2}$ and

$$1 \le (\cdots((p_1 - t_1)p_2 + t_1)\,p_3 - \cdots - t_n)\,p_{2n} + t_n.$$

Furthermore, if $1 < \alpha$

$\Longrightarrow\ \exists (A, B):\ 0 < B \le A,$

$$\Bigg\{A^{\frac{t_{n+2}}{2}}\Bigg(A^{-\frac{t_{n+1}}{2}}\Bigg(A^{\frac{t_n}{2}}\Bigg(A^{-\frac{t_n}{2}}\cdots \left(A^{\frac{t_1}{2}}\left(A^{-\frac{t_1}{2}}B^{p_1}A^{-\frac{t_1}{2}}\right)^{p_2}A^{\frac{t_1}{2}}\right)^{p_3} \cdots A^{-\frac{t_n}{2}}\Bigg)^{p_{2n}}A^{\frac{t_n}{2}}\Bigg)^{p_{2n+1}}A^{-\frac{t_{n+1}}{2}}\Bigg)^{p_{2n+2}}A^{\frac{t_{n+2}}{2}}\Bigg\}^{\frac{1-t_{n+1}+t_{n+2}}{\psi_1}\alpha}$$

$\not\le A^{(1-t_{n+1}+t_{n+2})\alpha},$

where

$$\psi_1 = (((\cdots((p_1 - t_1)p_2 + t_1)\,p_3 - \cdots - t_n)\,p_{2n} + t_n)\,p_{2n+1}$$

$$-t_{n+1})\,p_{2n+2} + t_{n+2}.$$

Theorem 5(KW).

Let $0 < p,\ \ 0 < s,\ \ 0 < t \leq 1,\ \ t \leq r.$

Furthermore, we assume (i) or (ii) of the following :

(i) $t < p$ and $\dfrac{1 - t + r}{(p - t)s + r} \cdot sp < 1$

(ii) $t = p < r$ and $p < 1$

$\Longrightarrow\ \exists(A, B) :\ 0 < B \leq A,$

$$\left\{A^{\frac{r}{2}}\left(A^{-\frac{t}{2}}\,B^p\,A^{-\frac{t}{2}}\right)^s A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)s+r}} \not\leq A^{1-t+r}.$$

Corollary.

Let $1 < p,\ 0 < s \leq \dfrac{1}{p},\ 0 < t \leq 1$ and $t \leq r.$

$\Longrightarrow\ \exists(A, B) :\ 0 < B \leq A,$

$$\left\{A^{\frac{r}{2}}\left(A^{-\frac{t}{2}}\,B^p\,A^{-\frac{t}{2}}\right)^s A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)s+r}} \not\leq A^{1-t+r}.$$

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